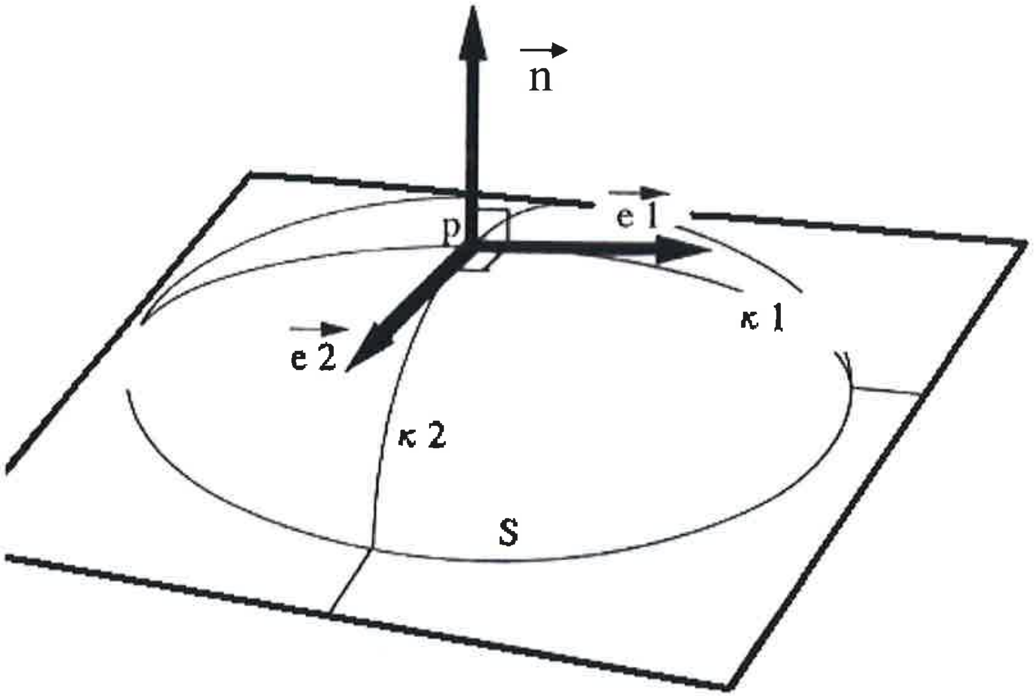
Principal Directions

- Viewpoint invariant differential features to represent intrinsic surface properties



Gaussian Curvature K＝k1・k2

Mean Curvature H＝(k1＋k2)/2

Min/Max Principal Curvatures

$$k1, k2 = H \pm \sqrt{H^2 - K}\,(k1 \geq k2)$$

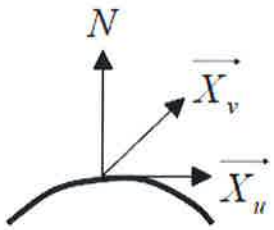
Min/Max Principal Directions

$$\overrightarrow{e1}, \overrightarrow{e2} \quad (\overrightarrow{e1} \times \overrightarrow{e2} = O)$$

Important formula

1. Surface $\overrightarrow{X}(u,v) = (x(u,v), y(u,v), z(u,v))$

$$= (x, y, z(x, y))$$



2. surface normal

$$N(u,v) = \frac{\overrightarrow{X_u} \times \overrightarrow{X_v}}{\left|\overrightarrow{X_u} \times \overrightarrow{X_v}\right|}$$

3. the first fundamental form

$$E = \overrightarrow{X_u} \cdot \overrightarrow{X_u} \qquad F = \overrightarrow{X_u} \cdot \overrightarrow{X_v} \qquad G = \overrightarrow{X_v} \cdot \overrightarrow{X_v}$$

4. the second fundamental form

$$e = \overrightarrow{N} \cdot \overrightarrow{X_{uu}} = -\overrightarrow{N_u} \cdot \overrightarrow{X_u}$$

$$f = \overrightarrow{N} \cdot \overrightarrow{X_{uv}} = -\overrightarrow{N_v} \cdot \overrightarrow{X_u} = -\overrightarrow{N_u} \cdot \overrightarrow{X_v}$$

$$g = \overrightarrow{N} \cdot \overrightarrow{X_{vv}} = -\overrightarrow{N_v} \cdot \overrightarrow{X_v}$$

$$\text{arc length} = \int_0^S \sqrt{E(du)^2 + 2F\,du\,dv + G(dv)^2}$$

$$\text{area} = \iint_\Omega \sqrt{EG - F^2}\,du\,dv$$

$$\text{Gaussian curvature} = \frac{eg - f^2}{EG - F^2}$$

$$\text{Mean curvature} = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$

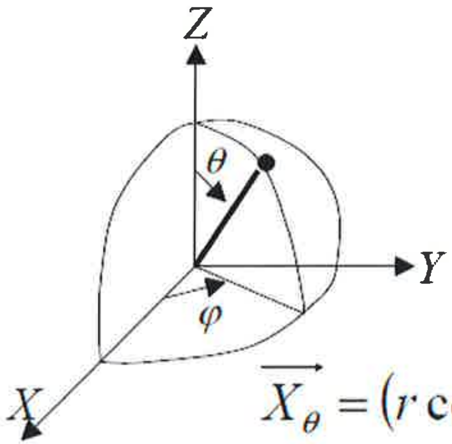
Gaussian Curvature $K = k1 \cdot k2$

Mean Curvature $H = (k1 + k2)/2$

Min/Max Principal Curvatures

$$k1, k2 = H \pm \sqrt{H^2 - K}\ (k1 \geq k2)$$

$$\vec{e1}, \vec{e2}\quad (\vec{e1} \times \vec{e2} = O)$$



$$\overrightarrow{X}(\theta,\varphi)=\left(X(\theta,\varphi),Y(\theta,\varphi),Z(\theta,\varphi)\right)$$
$$=\left(r\sin\theta\cos\varphi,r\sin\theta\sin\varphi,r\cos\theta\right)$$

$$\overrightarrow{X_\theta}=\left(r\cos\theta\cos\varphi,r\cos\theta\sin\varphi,-r\sin\theta\right)$$
$$\overrightarrow{X_\varphi}=\left(-r\sin\theta\sin\varphi,r\sin\theta\cos\varphi,0\right)$$

$$E=\overrightarrow{X_\theta}\cdot\overrightarrow{X_\theta}=r^2\cos^2\theta\cos^2\varphi+r^2\cos^2\theta\sin^2\varphi+r^2\sin^2\theta$$
$$=r^2\cos^2\theta+r^2\sin^2\theta=r^2$$
$$F=\overrightarrow{X_\theta}\cdot\overrightarrow{X_\varphi}=-r^2\sin\theta\cos\theta\sin\varphi\cos\varphi+r^2\sin\theta\cos\theta\sin\varphi\cos\varphi$$
$$=0$$
$$G=\overrightarrow{X_\varphi}\cdot\overrightarrow{X_\varphi}=r^2\sin^2\theta\sin^2\varphi+r^2\sin^2\theta\cos^2\varphi$$
$$=r^2\sin^2\theta$$

$$\vec{N} = \frac{\vec{X_\theta} \times \vec{X_\varphi}}{\left|\vec{X_\theta} \times \vec{X_\varphi}\right|}$$

$$|\mathbf{A}| = a_{11}\widetilde{a_{11}} + a_{12}\widetilde{a_{12}} + a_{13}\widetilde{a_{13}}$$

$$|\mathbf{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$\vec{X_\theta} \times \vec{X_\varphi} = \begin{pmatrix} i & j & k \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix}$$

$$= \left(r^2\sin\theta\right)\left\{(\sin\theta\cos\varphi)i + (\sin\theta\sin\varphi)j + (\cos\theta)k\right\}$$

$$\underline{\vec{N} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)}$$

$$\vec{X_{\theta\theta}} = (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, -r\cos\theta)$$

$$\vec{X_{\theta\varphi}} = (-r\sin\theta\sin\varphi, r\cos\theta\cos\varphi, -r\cos\theta)$$

$$\vec{X_{\varphi\varphi}} = (-r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi, 0)$$

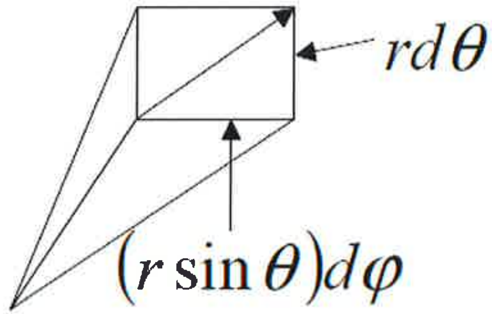
$$e = N\cdot\vec{X_{\theta\theta}} = -r\sin^2\theta\cos^2\varphi - r\sin^2\theta\sin^2\varphi - r\cos^2\theta = -r$$

$$f = N\cdot\vec{X_{\theta\varphi}} = -r\sin\theta\cos\theta\sin\varphi\cos\varphi + r\sin\theta\cos\theta\sin\varphi\cos\varphi = 0$$

$$g = N\cdot\vec{X_{\varphi\varphi}} = -r\sin^2\theta\cos^2\varphi - r\sin^2\theta\sin^2\varphi = -r\sin^{2\theta}$$

$$\text{arc lenght} = \int \sqrt{E(d\theta)^2 + 2F d\theta d\varphi + G(d\varphi)^2}$$

$$= \int \sqrt{r^2(d\theta)^2 + r^2 \sin^2\theta (d\varphi)^2}$$



$$\text{area} = \iint \sqrt{EG - F^2}\, d\theta d\varphi$$

$$= \iint (r^2 \sin\theta) d\theta d\varphi = \iint (r d\theta) \times (r \sin\theta) d\varphi$$

$$\text{GC} = \frac{eg - f^2}{EG - F^2} = \frac{r^2 \sin^2\theta}{r^4 \sin^2\theta} = \frac{1}{r^2} = \left(-\frac{1}{r}\right)\left(-\frac{1}{r}\right)$$

$$\text{MC} = \frac{1(eG - 2fF + gE)}{2(EG - F^2)} = \frac{1 - 2r^3 \sin^2\theta}{2(r^4 \sin^2\theta)} = -\frac{1}{r} = \frac{1}{2}\left\{\left(-\frac{1}{r}\right) + \left(-\frac{1}{r}\right)\right\}$$